

An Application of Second Derivative Backward Differentiation Formula Hybrid Block Method on Stiff Ordinary Differential Equations

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Abstract

In this paper, we developed a continuous scheme of four and five step with one off-grid point at collocation which provides the approximate solution of both linear and nonlinear stiff ordinary differential equations with constant step size. The continuous scheme is evaluated at both interpolation and collocation where necessary to give continuous hybrid block scheme and high order of accuracy with low error constants. Numerical results of the schemes are presented to compare with exact solutions and the results have shown that the (SDHBBDF) performed favorably when compare with existing methods.

Keywords: Collocation and interpolation, Hybrid block methods, Second Derivative and Stiff systems

1. Introduction

A first –order differential equation is an equation of the form

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

In which $f(x, y)$ is function two variables defined on a region in the xy – *plane* with initial condition $y(x_0) = y_0$ $a \leq x \leq b$ which is called initial value problems. The solution of equation (1) has been discussed by various researchers among them are Onumayi *et al* (1994) propose a linear multistep method of order 5 that is self-starting for the direct solution of the general second-order initial value problem (IVP). The method is derived by the interpolation and collocation of the assumed approximate solution and its second derivative. Some researchers have attempted the solution of directly using linear multistep methods without reduction to system of first order ordinary differential equations by Mohammed *et al* (2010). The block method produced numerical solutions with less than computational effort as to compare to non-block method by Majid (2004). Ebadi *et al.* (2010) worked on the hybrid BDF with one additional off-grid point introduced in the first derivative of the solution to improve the absolute stability region of the method. Block methods have been considered by various authors among who are A-stable implicit one block methods with higher orders by Shampine and Watts (1972). Hybrid blocks method of order six for the numerical solution of first order initial value problems. The method is based on collocation of the differential system and interpolation of the approximate at the grid and off-grid points by Areo and Adeniyi (2013).

In this paper, we developed four and five step of Second Derivative Hybrid Block Backward Differentiation Formula (SDHBBDF) for Numerical Solution of Stiff Ordinary Differential Equations.

2. The Derivation of the Method

We define a k-step SDHBBDF to numerical solution of (1) as

$$\sum_{j=0}^k \alpha_j(x) y_{n+j}(x) = h \beta_k(x) y_{n+k}(x) \quad (2)$$

Where $\alpha_j(x)$, $\beta_k(x)$ are the continuous coefficients and $k \in \mathbb{Z}^+$ is the step number of the method. And

$$\alpha_j(x) = \sum_{j=0}^k \alpha_j x^j, \quad j \in \{0, 1, \dots, k-1\} \quad (3)$$

$$\beta_k(x) = \sum_{j=0}^k \beta_k x^j, \quad j \in \{0, 1, \dots, k-1\} \quad (4)$$

$x_{n+j}: j = 0, 1, 2, \dots, t-1$ in (2) are ($0 \leq t \leq k$ arbitrary chosen interpolation points taken from $[x_n, \dots, x_{n+k}]$ and $x_j: j = 0, 1, \dots, m-1$ are m collocation points belonging to $[x_n, \dots, x_{n+k}]$. To get $\alpha_j(x)$, and $\beta_k(x)$, Sirisena *et al* (2004) arrived at a matrix equation of the form

$$DC = I \quad (5)$$

Where I is the identity matrix of dimension $(t+m) \times (t+m)$ while D and C are matrices defined as

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \dots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & \dots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & x_{n+t-1}^3 & x_{n+t-1}^4 & \dots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \dots & (t+m-1)x_n^{t+m-2} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \dots & (t+m-1)x_{n+1}^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+t-1} & 3x_{n+t-1}^2 & 4x_{n+t-1}^3 & \dots & (t+m-1)x_{n+t-1}^{t+m-2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & \dots & (t+m-1)(t+m-2)x_n^{t+m-3} \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & \dots & (t+m-1)(t+m-2)x_{n+1}^{t+m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & 6x_{n+t-1} & 12x_{n+t-1}^2 & \dots & (t+m-1)(t+m-2)x_{n+t-1}^{t+m-3} \end{pmatrix} \quad (6)$$

The above matrix (6) is the multistep collocation matrix of dimension $(t+m) \times (t+m)$ and

$$C = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & \dots & \alpha_{t-1,1} & h\beta_{0,1} & \dots & h\beta_{m-1,1} & h^2\gamma_{0,1} & \dots & h^2\gamma_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \dots & \alpha_{t-1,2} & h\beta_{0,2} & \dots & h\beta_{m-1,2} & h^2\gamma_{0,2} & \dots & h^2\gamma_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,j+m} & \alpha_{1,j+m} & \dots & \alpha_{t-1,j+m} & h\beta_{0,j+m} & \dots & h\beta_{m-1,j+m} & h^2\gamma_{0,j+m} & \dots & h^2\gamma_{m-1,j+m} \end{pmatrix} \quad (7)$$

Where t and m are defined as the number of interpolation and collocation points used respectively.

The columns of the matrix $C = D^{-1}$ give the continuous coefficients

$\alpha_j(x)$; $j = 0, 1, \dots, k-1$, $\beta_j(x)$; $j = 0, 1, \dots, k-1$ and $\gamma_j(x)$; $j = 0, 1, \dots, k-1$

The proposed four and five step of second derivative hybrid block backward differentiation formula (SDHBBDF) was developed subjected to the following conditions for matrix D :

For $k = 4$, evaluating (6) at $x_n = 0, h, 2h, \frac{7}{2}h$ and $4h$ and (2) becomes

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + h \left[\beta_3(x)f_{n+3} + \beta_{n+\frac{7}{2}}(x)f_{n+\frac{7}{2}} + \beta_4(x)f_{n+4} \right] + h^2[\gamma_2(x)g_{n+3} + \gamma_4(x)g_{n+4}] \quad (8)$$

Thus the matrix D in (6) becomes

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 \\ 0 & 1 & 2x_{n+7/2} & 3x_{n+7/2}^2 & 4x_{n+7/2}^3 & 5x_{n+7/2}^4 & 6x_{n+7/2}^5 & 7x_{n+7/2}^6 & 8x_{n+7/2}^7 \\ 0 & 1 & 6x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 \end{pmatrix} \quad (9)$$

Using Maple software, the inverse of the matrix in (9) is obtained and this yields the elements of the matrix D. The element of the matrix D substituted into (8) yields the continuous formulation of the method as:

$$\begin{aligned} y_n - \frac{43889121}{5338519} y_{n+1} + \frac{364359465}{5338519} y_{n+2} - \frac{325808863}{5338519} y_{n+3} \\ = \frac{6h}{5338519} [43986619f_{n+3} - 131818725f_{n+\frac{7}{2}} + 39066003f_{n+4} \\ + \frac{6h^2}{5338519} [100933g_n + 5360461g_{n+1} - 7124706g_{n+4}] \\ y_n + \frac{3870639}{528139} y_{n+1} - \frac{49623435}{528139} y_{n+2} + \frac{45224657}{528139} y_{n+3} \\ = -\frac{6h}{528139} [4800101f_{n+3} - 16213248f_{n+\frac{7}{2}} + 4696857f_{n+4}] \\ + \frac{h^2}{528139} [1211195g_{n+1} - 41449734g_{n+3} + 5086122g_{n+4}] \\ y_n - \frac{916731}{24929} y_{n+1} - \frac{6327585}{24929} y_{n+2} + \frac{28091}{97} y_{n+3} \\ = -\frac{6h}{24929} [685831f_{n+3} - 2803968f_{n+\frac{7}{2}} + 770427f_{n+4} \\ - \frac{18h^2}{24929} [86514g_{n+2} + 440719g_{n+3} - 45321g_{n+4}] \\ y_{n+\frac{7}{2}} = -\frac{4675}{354361344} y_n + \frac{35973}{118120448} y_{n+1} - \frac{865375}{118120448} y_{n+2} + \frac{356854225}{354361344} y_{n+3} \\ + \frac{35h}{59060224} [510455f_{n+3} + 350976f_{n+\frac{7}{2}} - 29115f_{n+4} + \frac{525h^2}{59060224} [4363g_{n+3} + 291g_{n+4}]] \end{aligned} \quad (10)$$

$$\begin{aligned} y_{n+4} = \frac{1}{100933} y_n - \frac{20}{100933} y_{n+1} + \frac{324}{100933} y_{n+2} + \frac{100628}{100933} y_{n+3} + \frac{12h}{100933} [1906f_{n+3} + \\ 4608f_{n+\frac{7}{2}} + 1921f_{n+4} + \frac{72h^2}{100933} [17g_n - 22g_{n+4}]] \end{aligned} \quad (11)$$

For $k = 5$, evaluating (6) at $x_n = 0, h, 2h, 3h, \frac{9}{2}h$ and $5h$ and (2) becomes

$$\begin{aligned} y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_4(x)y_{n+4} \\ + h \left[\beta_4(x)f_{n+4} + \beta_{n+\frac{9}{2}}(x)f_{n+\frac{9}{2}} + \beta_5(x)f_{n+5} \right] \\ + h^2 [\gamma_4(x)g_{n+4} + \gamma_5(x)g_{n+5}] \end{aligned} \quad (12)$$

Thus the matrix D in (6) becomes

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 & 9x_{n+4}^8 \\ 0 & 1 & 2x_{n+9/2} & 3x_{n+9/2}^2 & 4x_{n+9/2}^3 & 5x_{n+9/2}^4 & 6x_{n+9/2}^5 & 7x_{n+9/2}^6 & 8x_{n+9/2}^7 & 9x_{n+9/2}^8 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 & 8x_{n+5}^7 & 9x_{n+5}^8 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 & 72x_{n+5}^7 \end{pmatrix}$$

(13)

Similarly, we generate the continuous formulation of the new method as:

$$\begin{aligned} y_n - \frac{58246497728}{8106149141} y_{n+1} + \frac{287889150672}{8106149141} y_{n+2} - \frac{80444045248}{8106149141} y_{n+3} \\ + \frac{1612464238619}{8106149141} y_{n+4} + \frac{12h}{8106149141} [60796088575 f_{n+4} \\ - 247880384512 f_{n+\frac{9}{2}} + 69021826272 f_{n+5}] \\ + \frac{72h^2}{40530745705} [66359537 g_n - 93948417580 g_{n+4} \\ + 10159234608 g_{n+5}] \\ y_n + \frac{3679563944}{743599935} y_{n+1} - \frac{3572959464}{82622215} y_{n+2} + \frac{4537703608}{16524443} y_{n+3} - \frac{176463191063}{743599935} y_{n+4} \\ = \frac{4h}{24766645} [5486226727 f_{n+4} - 25244983296 f_{n+\frac{9}{2}} + 6934049604 f_{n+5}] \\ + \frac{8h^2}{24766645} [66359537 g_{n+1} + 5898605983 g_{n+4} - 608582079 g_{n+5}] \\ y_n - \frac{2380511872}{71172879} y_{n+1} - \frac{736521408}{23724293} y_{n+2} + \frac{19444153216}{23724293} y_{n+3} - \frac{53813556431}{71172879} y_{n+4} \\ = \frac{4h}{23724293} [1300871131 f_{n+4} - 7145226240 f_{n+\frac{9}{2}} + 1922981184 f_{n+5}] \\ - \frac{8h^2}{23724293} [132719074 g_{n+2} - 1733268759 g_{n+4} + 167226176 g_{n+5}] \\ y_n - \frac{414537320}{18774287} y_{n+1} + \frac{110394952}{2682941} y_{n+2} + \frac{19790816104}{18774287} y_{n+3} - \frac{27122653735}{18774287} y_{n+4} \\ = \frac{12h}{18774287} [782440151 f_{n+4} - 4864139264 f_{n+\frac{9}{2}} + 1241907228 f_{n+5}] \\ + \frac{72h^2}{18774287} [132719074 g_{n+3} + 429136117 g_{n+4} - 35225544 g_{n+5}] \end{aligned}$$

$$\begin{aligned}
y_{n+\frac{9}{2}} &= \frac{6770575}{2174469308416} y_n - \frac{7688475}{135904331776} y_{n+1} + \frac{4798521}{7346180096} y_{n+2} \\
&\quad - \frac{1424749725}{135904331776} y_{n+4} \\
&\quad + \frac{315h}{543617327104} \left[526591065 f_{n+4} - 346107904 f_{n+\frac{9}{2}} - 25923240 f_{n+5} \right] \\
&\quad + \frac{99225h^2}{271808663552} [114319 g_{n+4} + 5964 g_{n+5}] \\
y_{n+5} &= -\frac{213}{66359537} y_n + \frac{3625}{66359537} y_{n+1} - \frac{1000}{1793501} y_{n+2} + \frac{429000}{66359537} y_{n+3} \\
&\quad + \frac{65964125}{66359537} y_{n+4} \\
&\quad + \frac{60h}{66359537} \left[247625 f_{n+4} + 614400 f_{n+\frac{9}{2}} + 250051 f_{n+5} \right] \\
&\quad + \frac{1800h}{66359537} [335 g_{n+4} - 563 g_{n+5}]
\end{aligned} \tag{14}$$

3. Analysis of the New Methods

We consider the analysis of the newly constructed methods such as order, error constant, consistency, convergence and the regions of absolute stability of the methods. Following (Fatunla, 1991) and (Lambert, 1973) we defined the local truncation error associated with (11) and (14) to be linear difference operator

$$L[y(t); h] = \sum_{j=0}^k \alpha_j y_{n+j} - h \beta_k f_{n+k} - h^2 \gamma_k g_{n+k} \tag{15}$$

Assuming that $y(t)$ is sufficiently differentiable, we can expand the terms in (15) as a Taylor series and comparing the coefficients of h gives

$$L[y(t); h] = c_0 y(t) + c_1 h y'(t) + c_2 h^2 y''(t) + \dots + c_p h^p y^{(p)}(t) + \dots \tag{16}$$

Where the constant coefficients $C_p, p = 0, 1, 2, \dots, j = 1, 2, \dots, k$ are given as follows:

$$\begin{aligned}
C_0 &= \sum_{j=0}^k \alpha_j \\
C_1 &= \sum_{j=1}^k j \alpha_j, \\
&\vdots \\
C_q &= \left[\frac{1}{q!} \sum_{j=0}^k j^q \alpha_j - q(q-1) \sum_{j=1}^k j^{q-2} \beta_j \right]
\end{aligned}$$

According to (Henrici, 1962), methods (8) and (12) have order p if

$C_0 = C_1 = C_2 \dots C_p = 0$ and $C_{p+1} \neq 0$. Therefore, C_{p+1} is the error constant and $C_{p+1} h^{p+1} y^{(p+1)}(t_n)$ is the principal local truncation error at point t_n . It was established from the evaluation that block methods (11) and (14) have order and error constants.

4. CONVERGENCE

The convergence of the new block methods is determined using the approach by Fatunla (1991) and Chollom et al (2007) for linear multistep methods, where the block methods are represented in single block, r point multistep method of the form

$$A^{(0)} y_{m+1} = \sum_{i=1}^k A^{(i)} y_{m+i} + h \sum_{i=0}^k B^{(i)} f_{m+i} \tag{17}$$

Where h is a fixed mesh size within a block, $A^i, B^i, i = 0, 1, 2, \dots, k$ are $r \times r$ identity while y_m, y_{m-1} and y_{m+1} are vectors of numerical estimates.

4.1 Definition: A numerical method is said to be A-stable if the whole of the left-half plane $\{Z: \text{Re}(Z) \leq 0\}$ is contained in the region. $\{Z: \text{Re}(Z) \leq 1\}$ Where $R(Z)$ is called the stability polynomial of the method (Lambert, 1973)

4.2 Definition: A numerical method is said to be $A(\alpha)$ -stable, $\alpha \in (0, \frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge $W_\alpha = \{h\lambda \mid -\alpha < \pi - \arg h\lambda < \alpha\}$ (Lambert, 1973),

The block method (11) expressed in the form of (17) gives the characteristic polynomial of the block method

$$\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)}) \quad (18)$$

$$= \lambda^4(\lambda - 1) = 0 \quad (19)$$

Therefore, $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. The block method (17) by definition is $A(\alpha)$ -stable and by Henrici (1962), the block method is convergent.

Similarly, The block method (14) expressed in the form of (17) gives the characteristic polynomial of the block method (18)

$$= \lambda^5(\lambda - 1) = 0 \quad (20)$$

Therefore, $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$. The block method (17) by definition is

$A(\alpha)$ -stable and by Henrici (1962), the block method is convergent.

5. Numerical experiments

Problem 1: Consider the Stiff nonlinear system of two dimensional Kaps problem with corresponding initial

$$\text{conditions} \begin{bmatrix} y'_1(x) \\ y'_2(x) \end{bmatrix} = \begin{bmatrix} -1002y_1(x) + 1000y_2^2(x) \\ y_1(x) - y_2(x) + y_2^2(x) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The analytic solution is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \exp(-2x) \\ \exp(-x) \end{bmatrix}$$

Problem 2: A two-dimensional SODEs is considered (Wu, X.Y and Xiu, J.L 2001)

$$y'_1 = -500000.5y_1 + 499999.5y_2 \quad y_1(0) = 0$$

$$y'_2 = 499999.5y_1 - 500000.5y_2 \quad y_2(0) = 2$$

The exact solutions are: $y_1(x) = -e^{\lambda_1 x} + e^{\lambda_2 x}$

$$y_2(x) = e^{\lambda_1 x} + e^{\lambda_2 x} \quad \lambda_1 = -10^6, \lambda_2 = -1$$

6. Conclusion

There are different hybrid block methods for solving stiff ordinary differential equations with components of both grid and off-grid points. In this paper, we proposed high order block hybrid k -step (SDHBBDF) with $k = 4$ and 5 for numerical solution of both linear and nonlinear stiff systems. The proposed methods give good approximate solution and reduced computational cost as shown in figure 3,4,5 and 6. Moreover, the schemes are $A(\alpha)$ -stable, consistent and convergent.

7. References

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Table 1: Order and Error Constants for the block methods (SDHBBDF Case $k = 4$)

Order	Error constants
8	$\frac{16524443}{2391656512}$
8	$-\frac{23724293}{3549094080}$
8	$-\frac{2682041}{167522880}$
8	$-\frac{27635}{68037378048}$
8	$-\frac{71}{169567440}$

Table 2: Order and Error Constants for the block methods (SDHBBDF Case $k = 5$)

Order	Error Constants
8	$-\frac{5946222278}{607961185575}$
8	$\frac{1173242432}{130129988625}$
8	$\frac{40962556}{1966619025}$
8	$\frac{341646346}{9856500675}$
8	$-\frac{3026919}{17395754467328}$
8	$\frac{310}{1393550277}$

The stability polynomial of the methods which is plotted in MATLAB environment to produce the required absolute stability region of the methods as shown in figures 1 and 2.

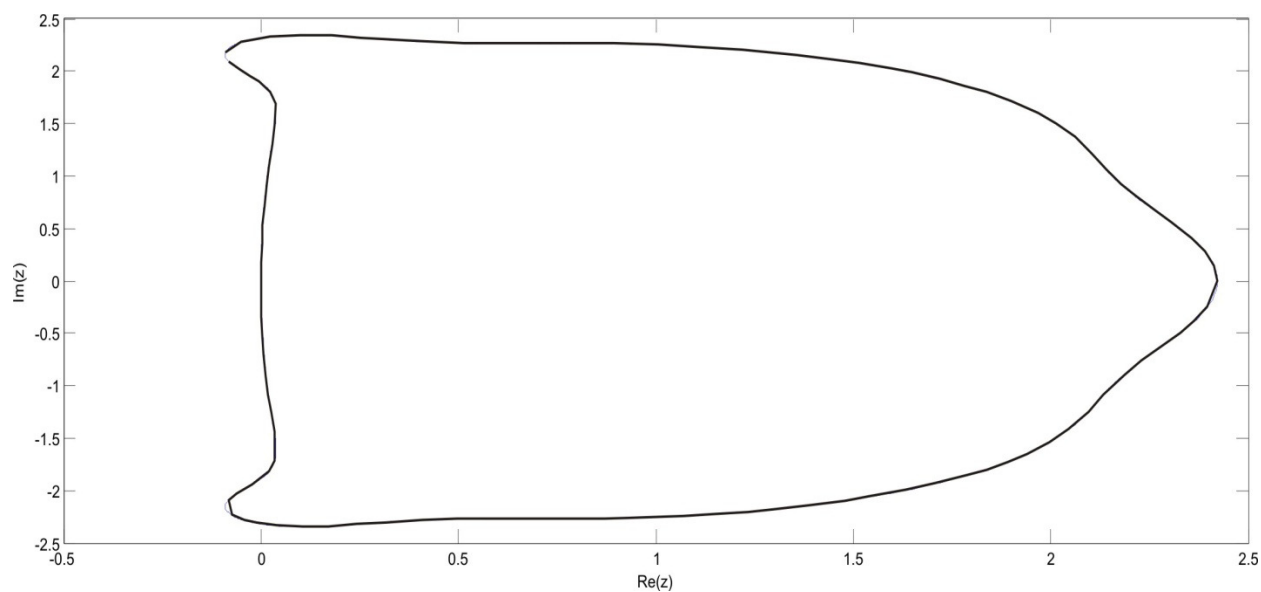


Fig.1 Absolute Stability Regions for the hybrid block methods (SDHBBDF Case $k = 4$)

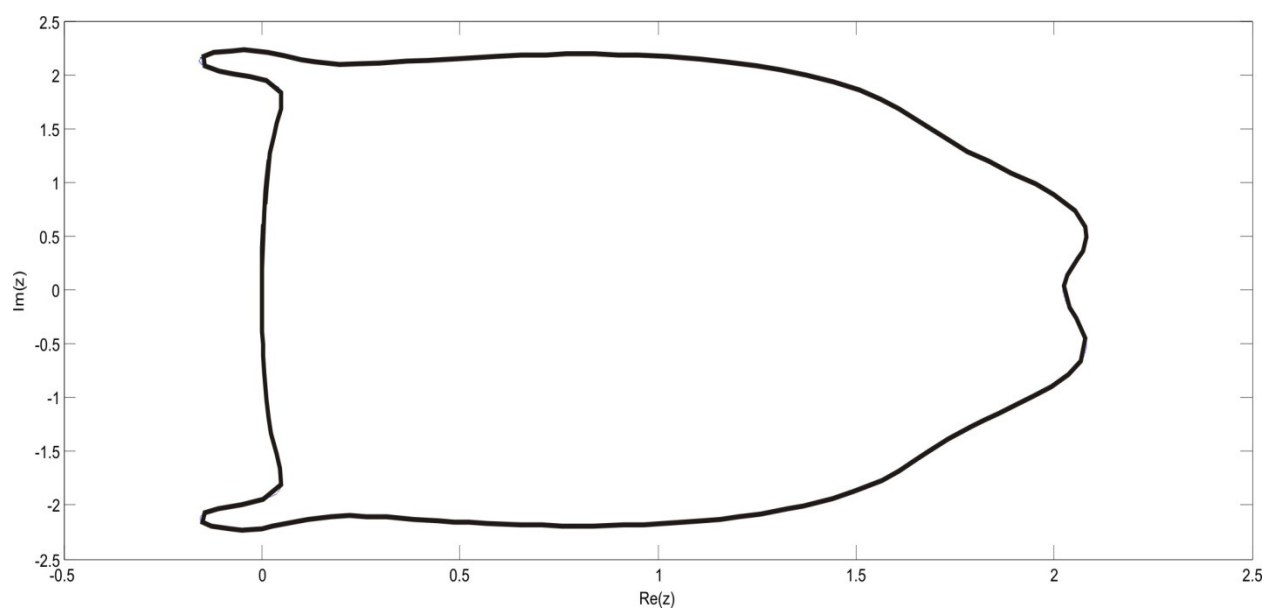


Fig.2 Absolute Stability Regions for the hybrid block methods (SDHBBDF Case $k = 5$)
Some numerical results to illustrate the performance of the methods executed in Matlab language

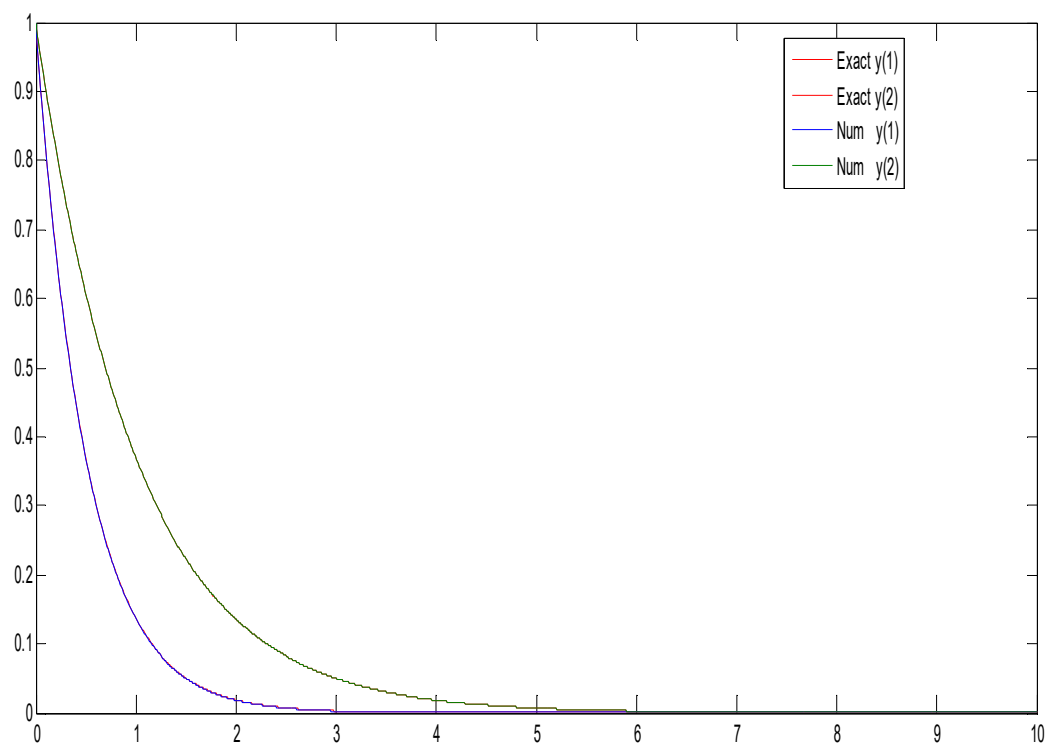


Fig.3 Problem 1: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 4$)

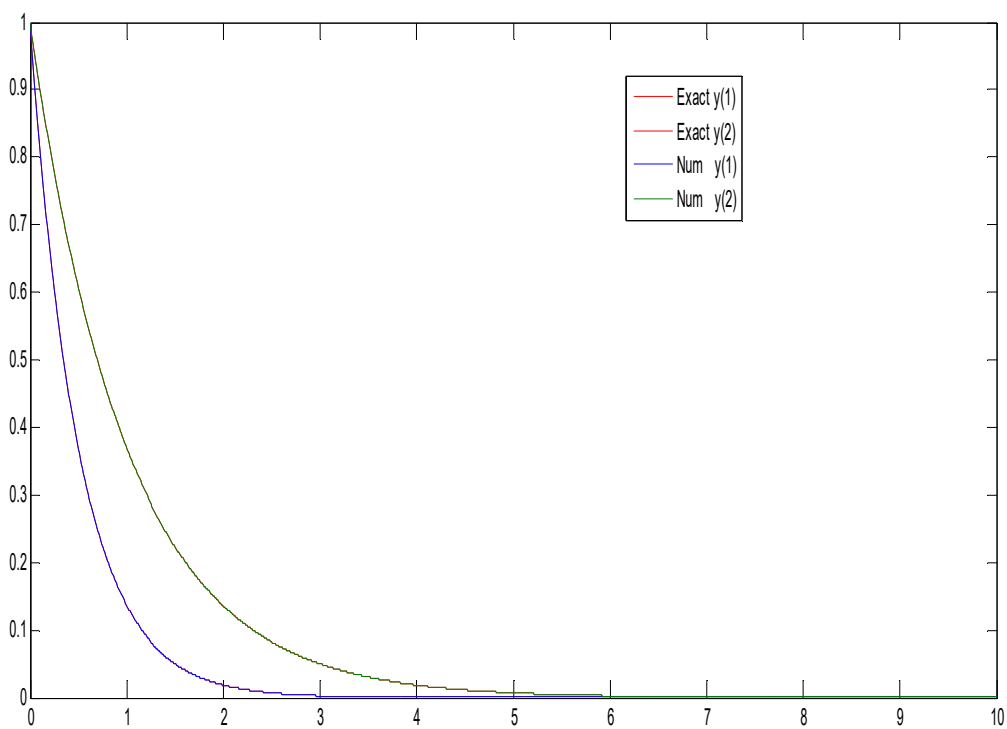


Fig.4 Problem 1: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 5$)

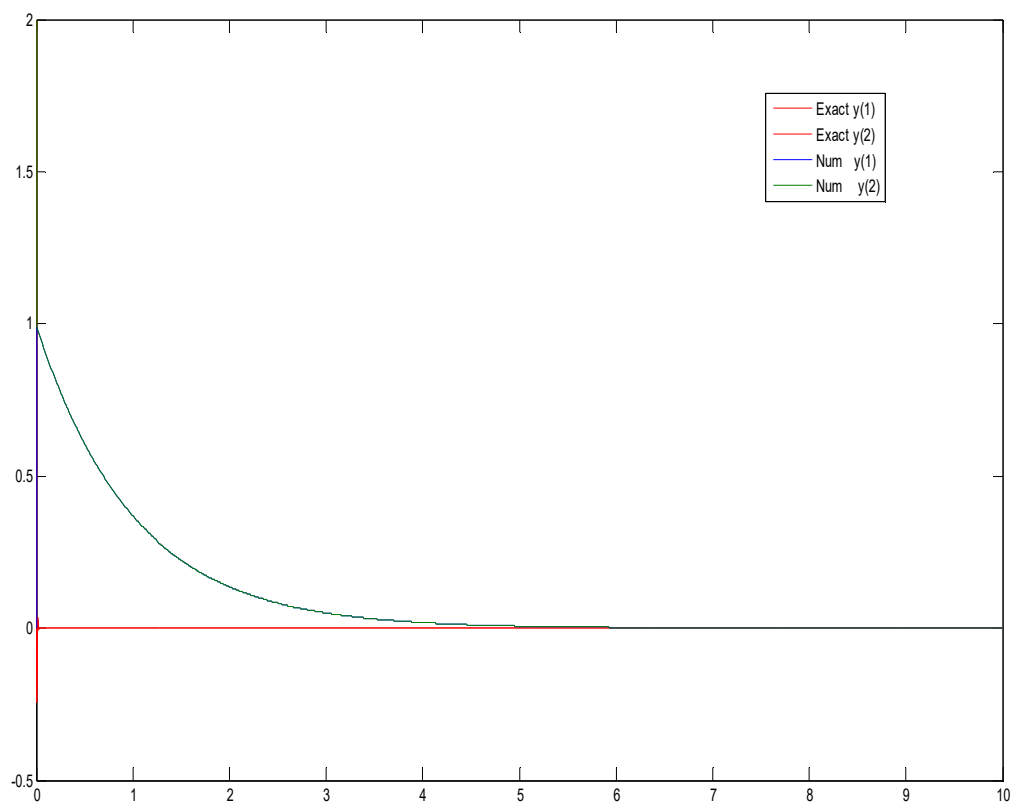


Fig.5 Problem 2: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 4$)

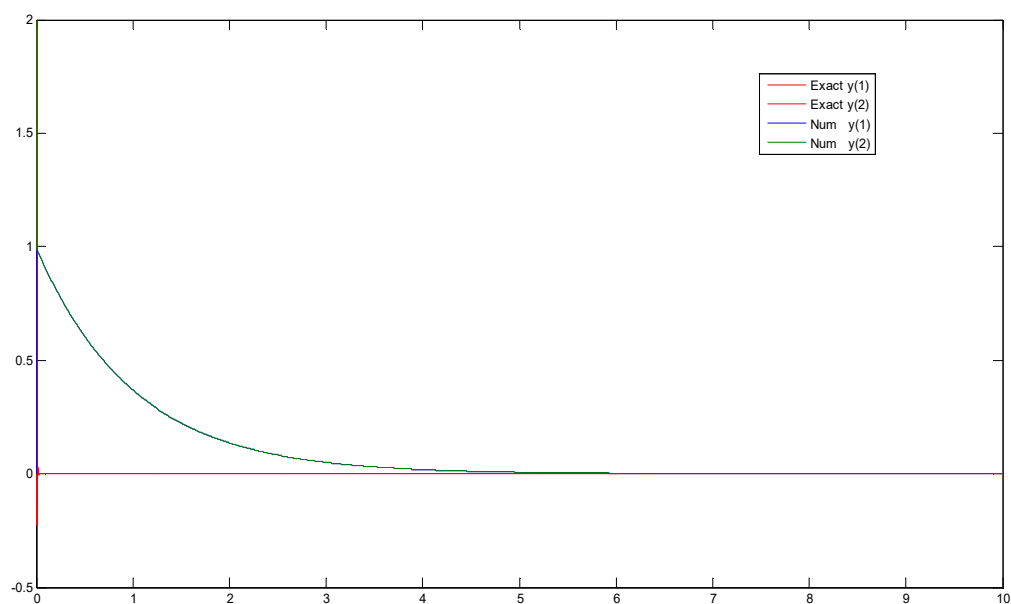


Fig.6 Problem 2: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 5$)